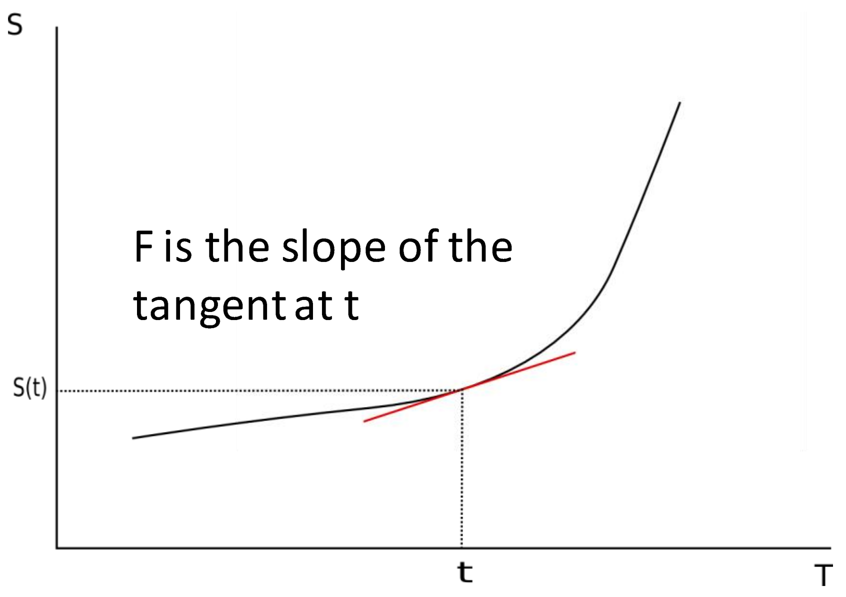
System Dynamics

- System dynamics deals with
- *Stocks* – how much there is of a given quantity
- *Flows* – rates of change of the stocks
- *Variables* – that weight flows
- *Links* – that connect variables to stock or flows

- Since flows can link together different stocks, systems that include complex feedback loops can be modelled

System Dynamics

- The Stock-flow model is essentially a diagrammatic way of representing differential equations
- Rate of change of stock $dS/dT = F(S,T)$



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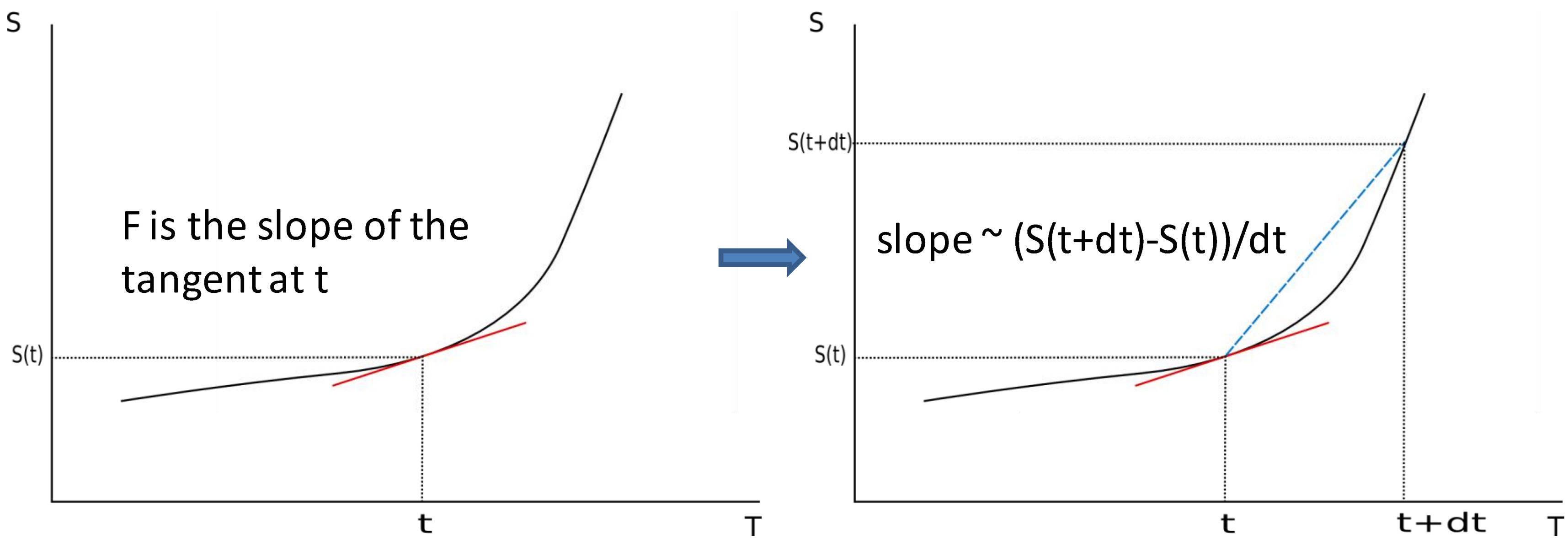

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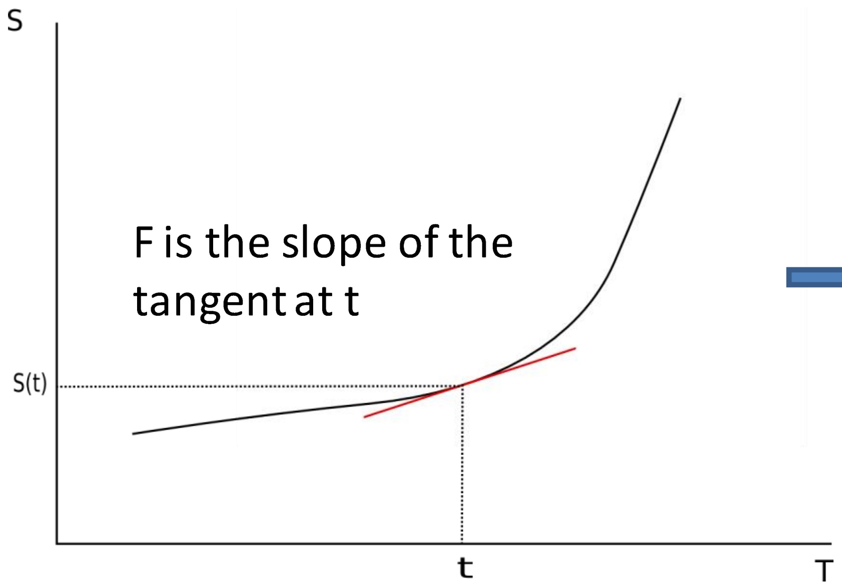

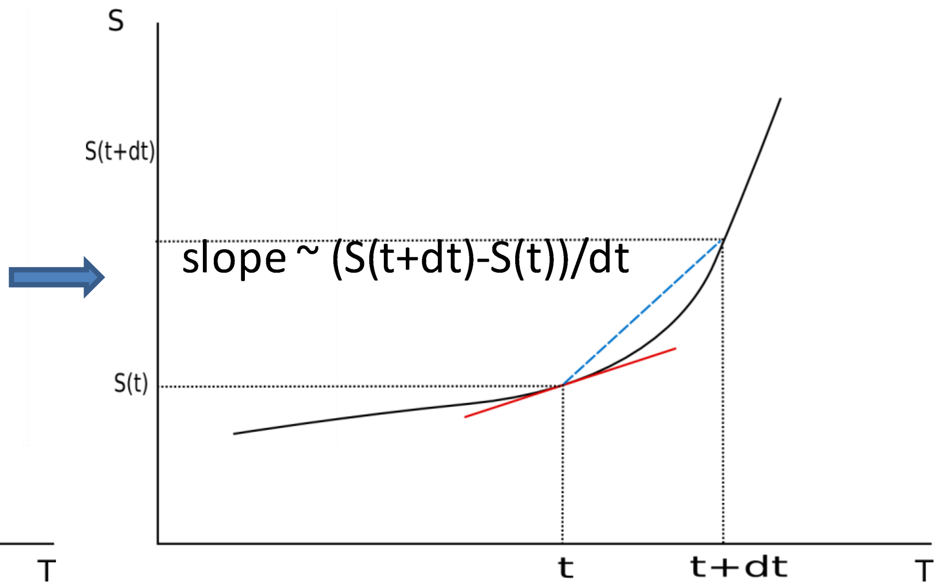

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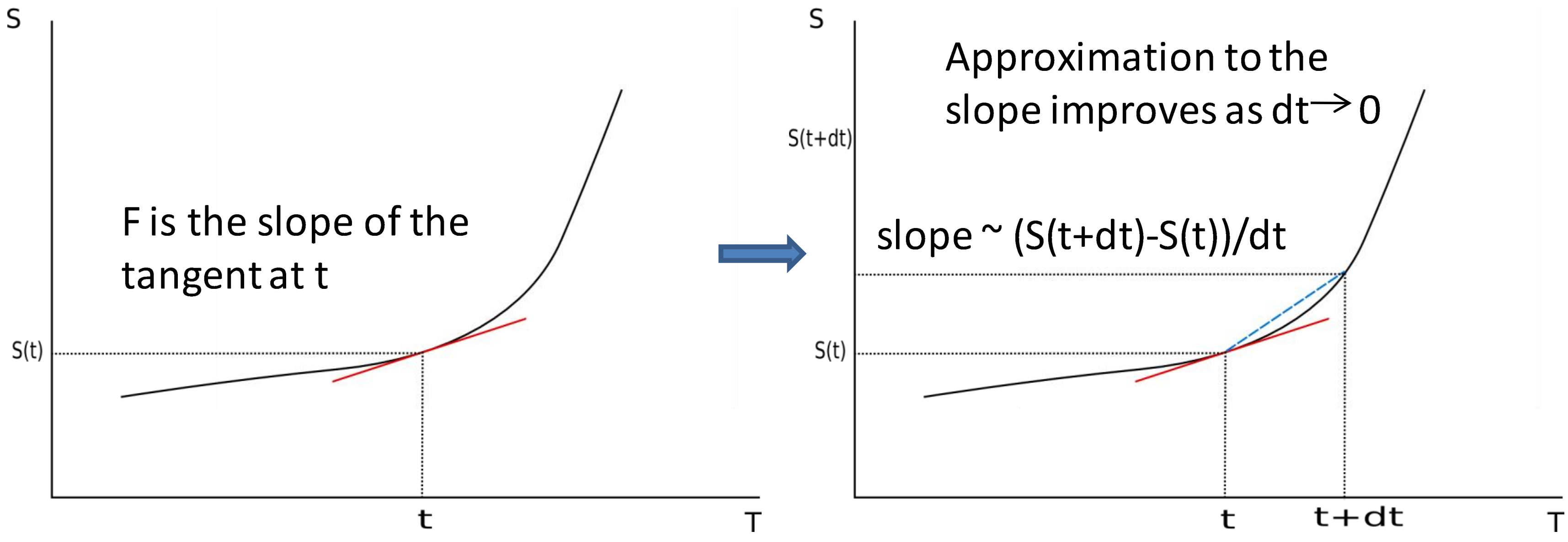


System Dynamics

- The Stock-flow model is essentially a diagrammatic way of representing differential equations
- Rate of change of stock dS/dT = F(S,T)

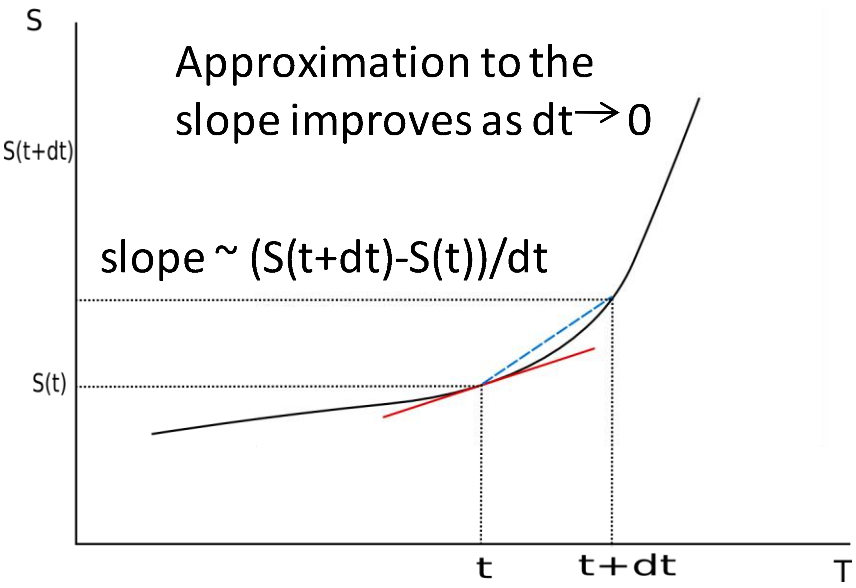
Computers need to use a finite dt so:-

We replace the differential equation

$$\frac{dS}{dt} = F(S,T)$$

with the difference equation

$$\frac{S(t+\delta t) - S(t)}{\delta t} = F(S,T)$$



System Dynamics

- Re-arranging we get

$$S(t+\delta t) = S(t) + F(S,T) \times \delta t$$

- This is called the forward Euler method
  - Exact only for straight lines
  - Accuracy improves as dt $\rightarrow$ 0
  - Large dt may not just be inaccurate but unstable
    - Results are not just inaccurate but disastrous!

Euler method example

Rate of change of population depends on its current value

Differential equation

$$\frac{dN}{dt} = \lambda N$$

Exponential growth (λ>0) or decay (λ<0)

$$N = N(0)\exp(\lambda t)$$

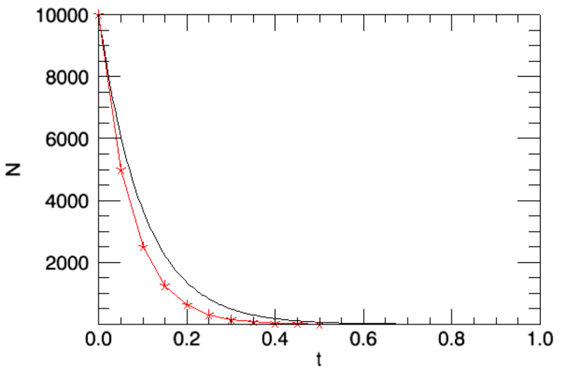
Difference equation

$$N(t+\delta t) = N(t)(1+\lambda\delta t) = N(t)(1+k)$$

Geometric growth (k >0) or decay (-1<k<0)
Oscillatory growth (k < -2)or decay(-2<k<-1)

$$N(n) = (1+\lambda\delta t)^n N(0) = (1+\frac{\lambda t}{n})^n N(0)$$

$$t = n\delta t$$



λ= -10 , dt=0.05

Differential equation in black
Difference equation in red

Note the error from a finite timestep

(but which equation is the better approximation?)

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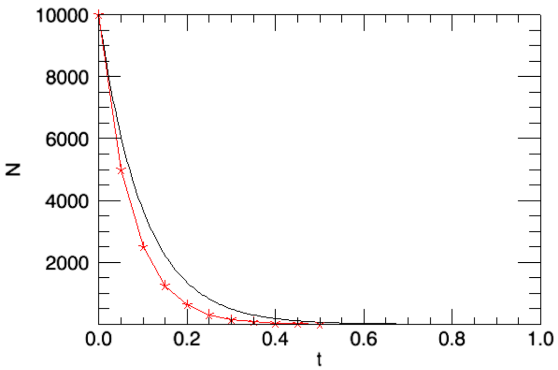

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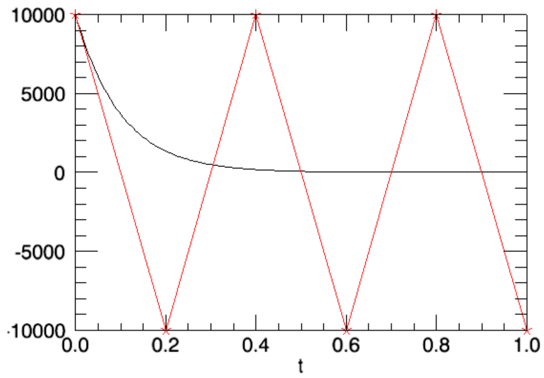

λ= -10 , dt=0.2

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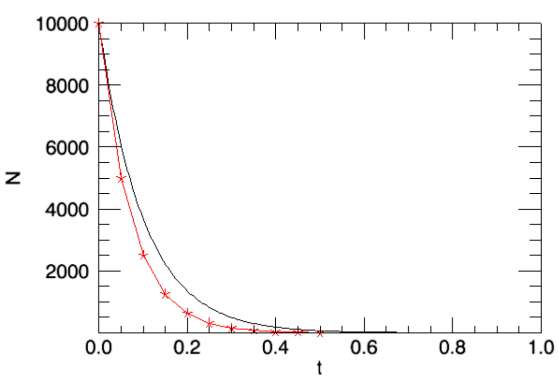

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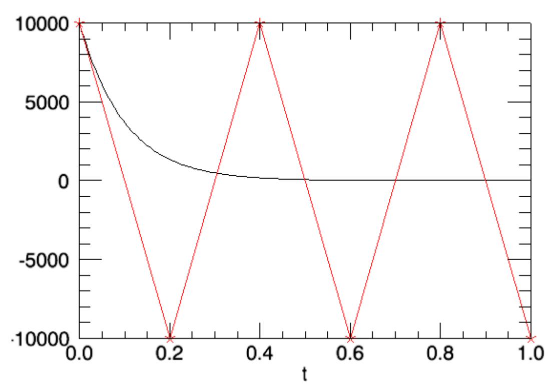

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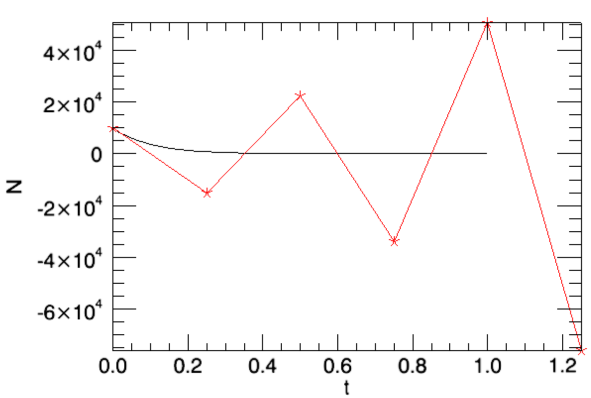

λ= -10 , dt=0.25

Euler method example

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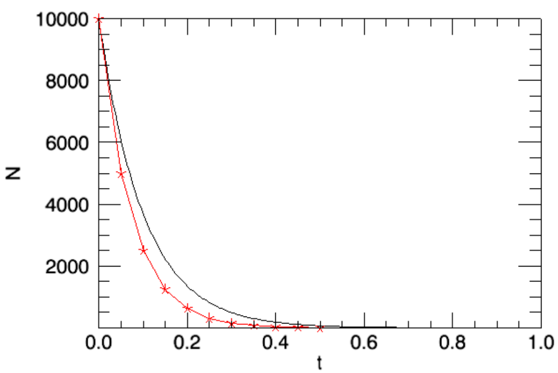

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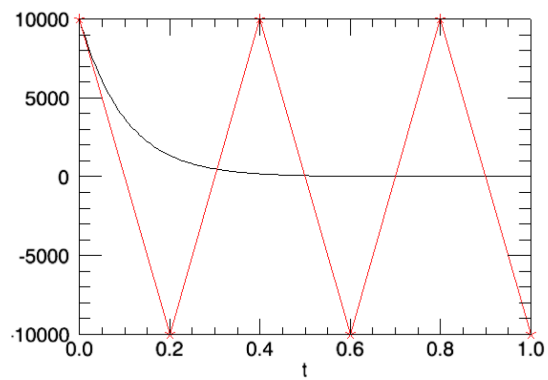

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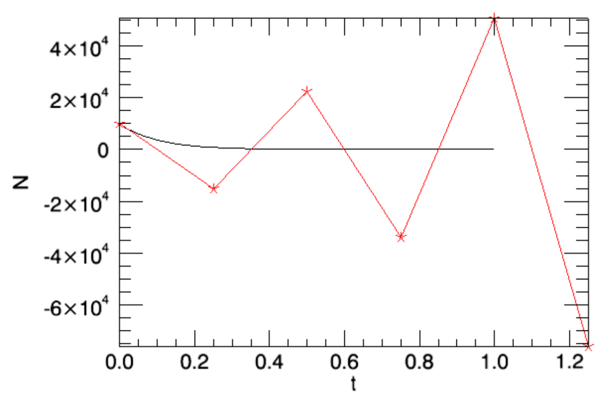


λ= -10 , dt=0.25

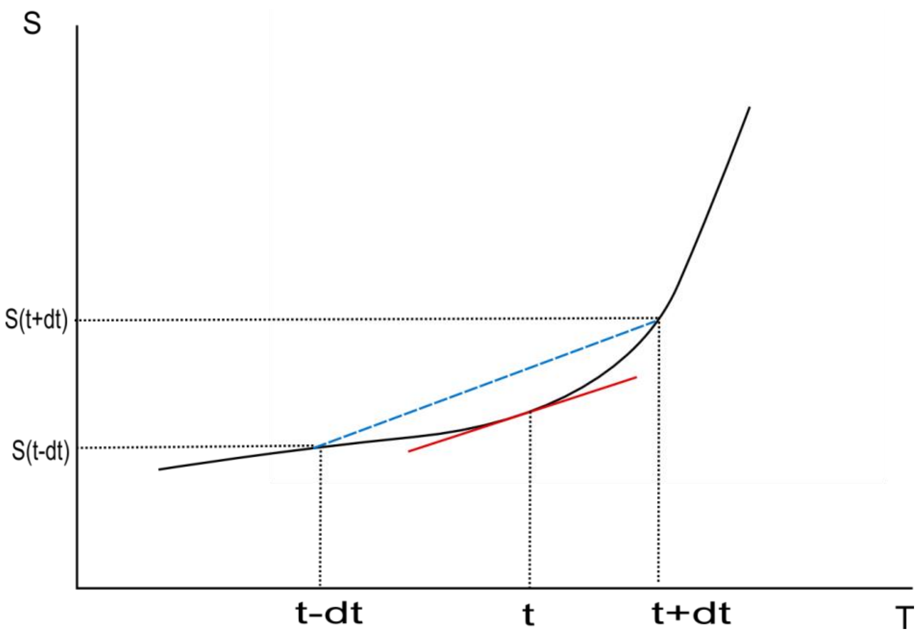
Kerboom!

Leapfrog method

Try to get a more accurate approximation good for quadratic curves

$$\frac{dS}{dt} \approx \frac{S(t+dt) - S(t-dt)}{2\delta t}$$

$$\frac{dN}{dt} = \lambda N$$



$$N(t+dt) = N(t-dt) + 2\delta t N(t)$$

Need two starting values
N(0) and N(dt)

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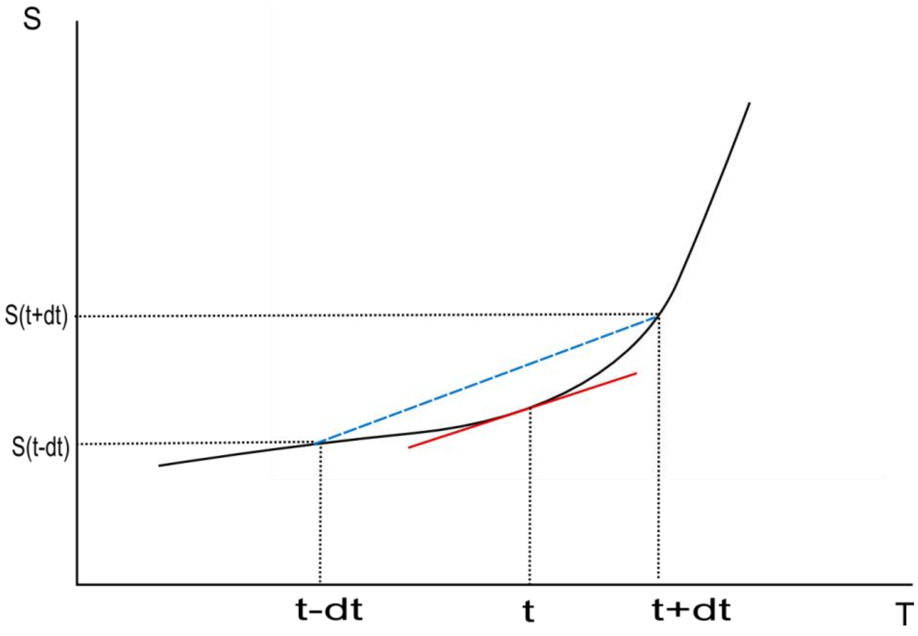

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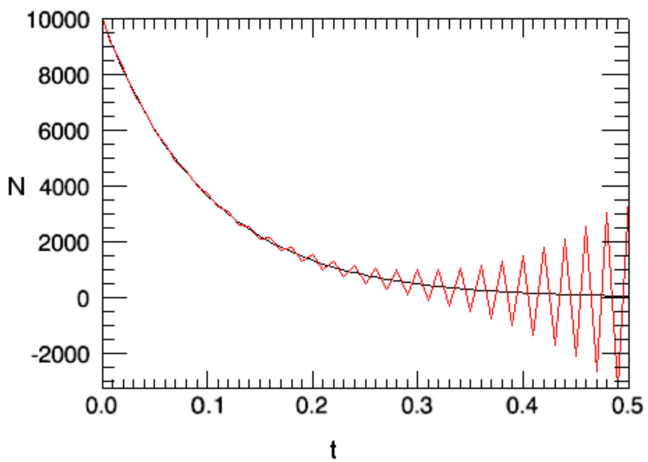

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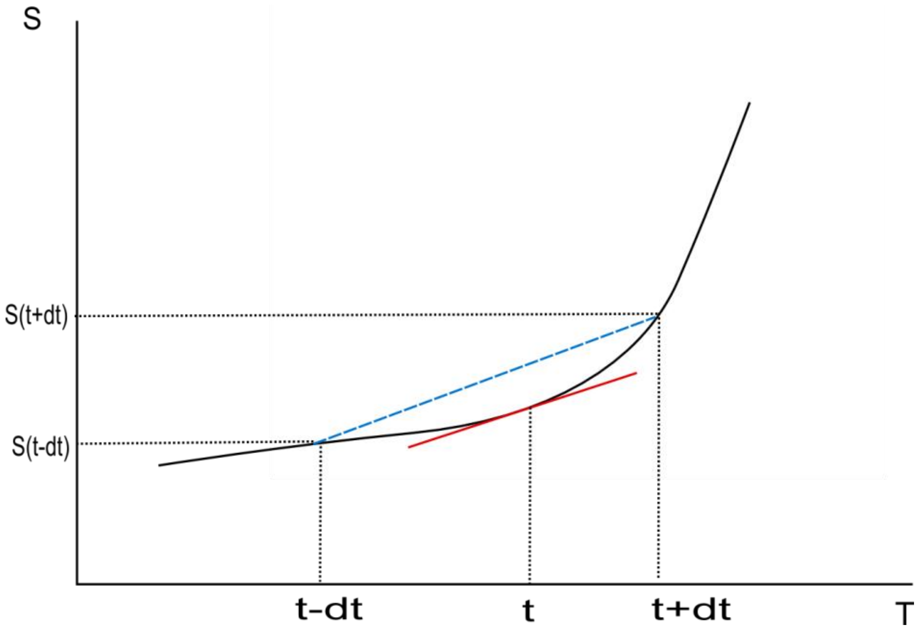

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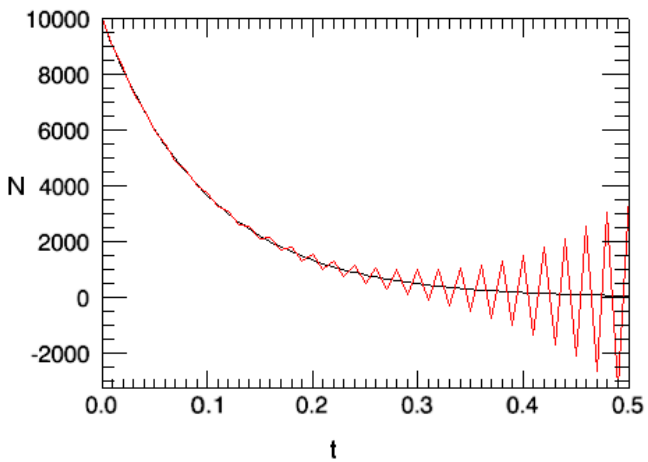

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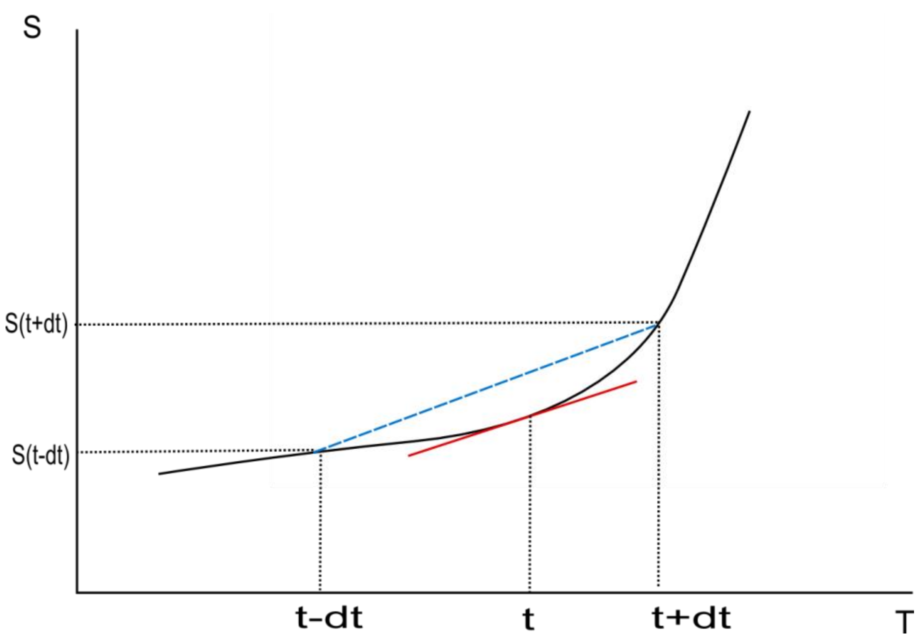

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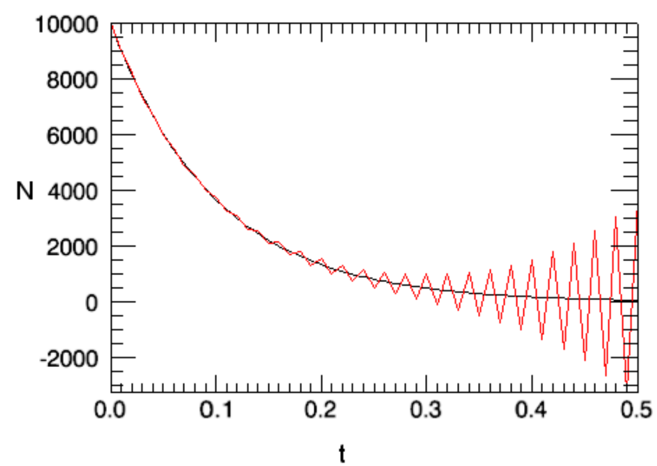


Higher order methods can be better,
But need care –
e.g. Midpoint method, Runge-Kutta methods

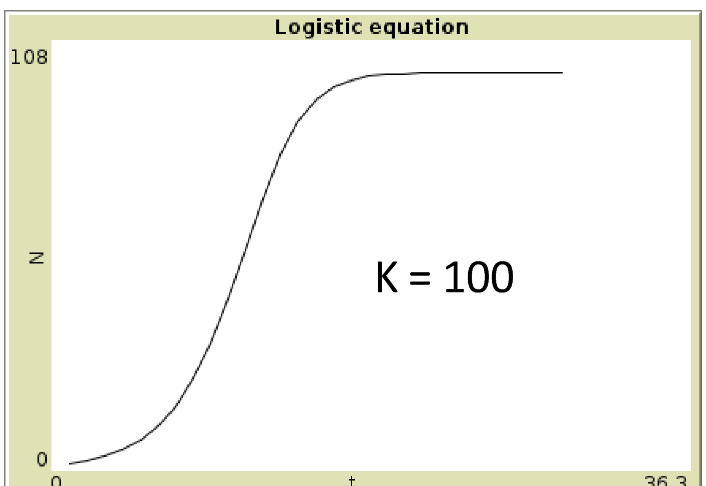
Initially more accurate than forward Euler,
but...unstable for all dt!

The logistic equation

The logistic equation
with carrying capacity K

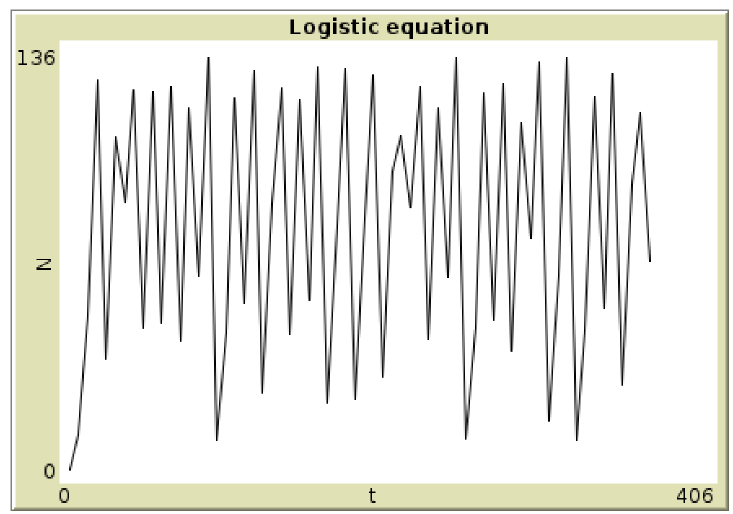
$$\frac{dN}{dt} = rN(1 - N/K)$$

$$N = \frac{KN(0)}{N(0) + (K - N(0))\exp(-rt)}$$



The discrete version can
exhibit deterministic chaos

$$N(t + \delta t) = N(t)(1 + r\delta t) - N(t)^2 r\delta t / K$$

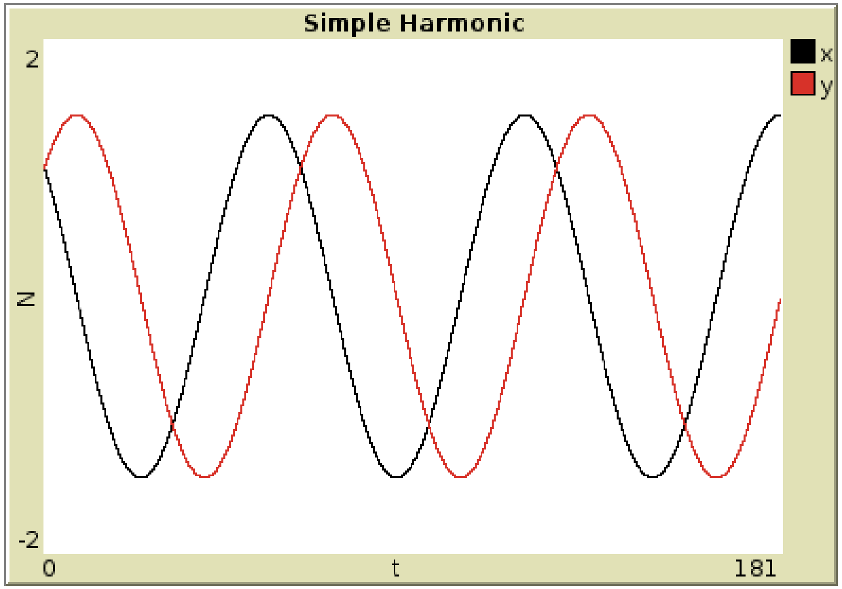


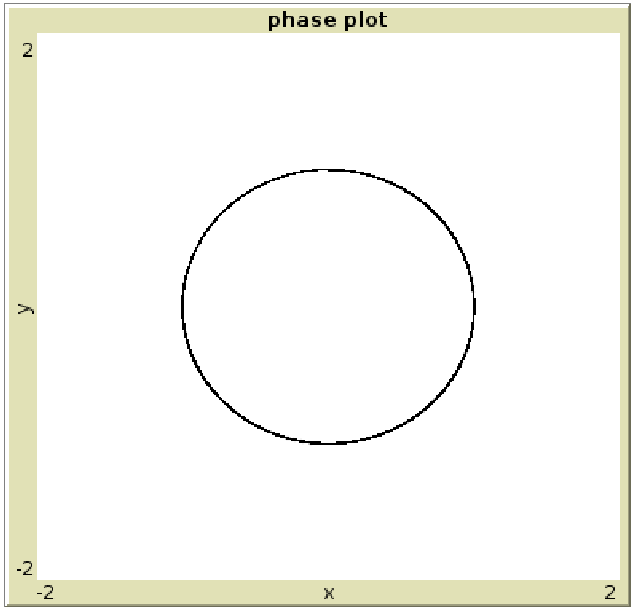
Coupled Equations

$$\frac{\partial x}{\partial t} = -\omega y$$

$$\frac{\partial y}{\partial t} = \omega x$$

Simple harmonic motion
Sinusoidal oscillations





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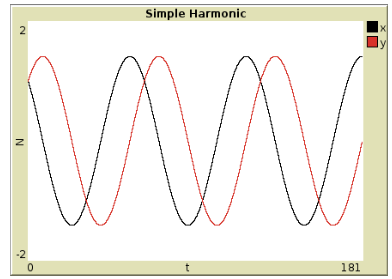
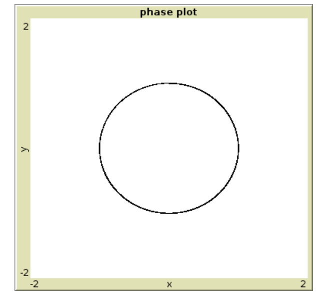
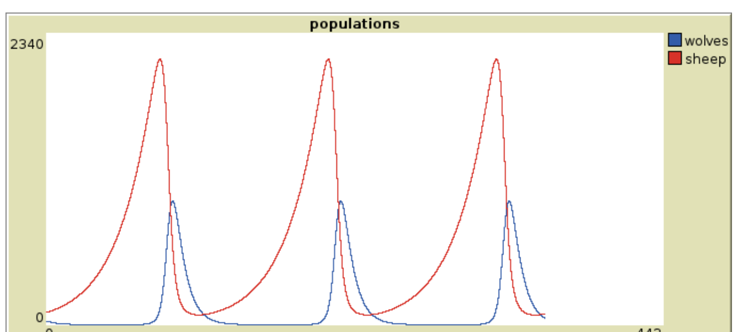


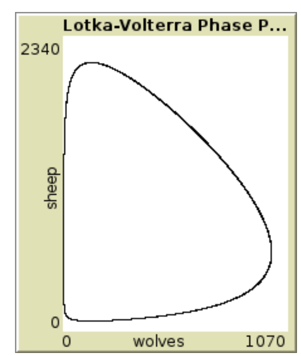


$$\frac{\partial x}{\partial t} = \alpha x - \beta xy$$

$$\frac{\partial y}{\partial t} = \gamma xy - \delta y$$

The Lotka-Volterra equations
Predator-prey
Non-linear oscillations





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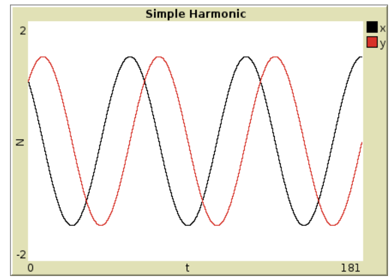

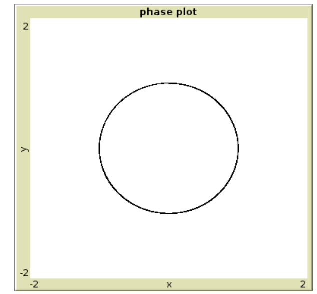

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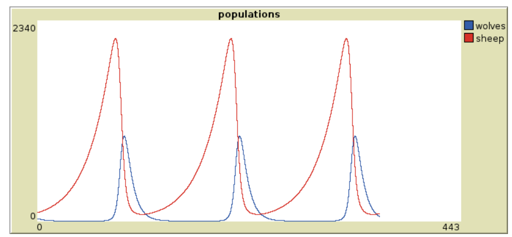
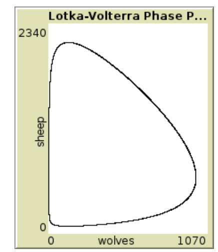


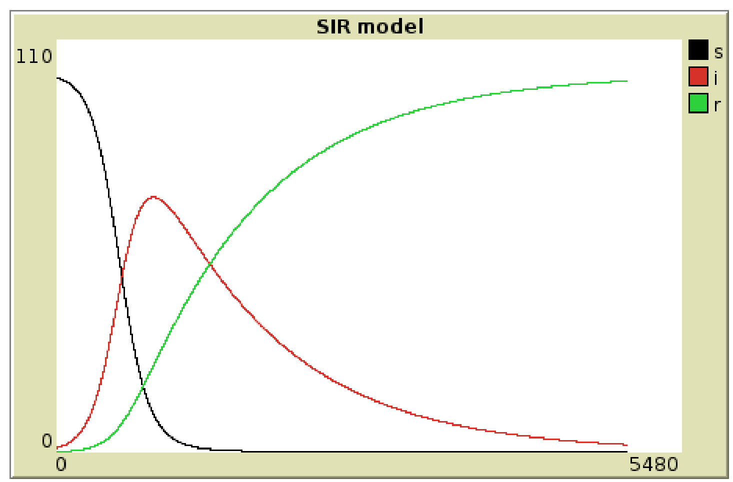


$$\frac{\partial S}{\partial t} = -\beta SI$$

$$\frac{\partial I}{\partial t} = \beta SI - \gamma I$$

$$\frac{\partial R}{\partial t} = \gamma I$$

The SIR equations
for spread of disease



Space and time

$$\frac{\partial q}{\partial t} = k\frac{\partial^2 q}{\partial x^2}$$

The diffusion equation in one dimension

Forward Euler in time

Discrete approximation to the second derivate

$$\frac{q(t+\delta t, x) - q(t, x)}{\partial t} = k\frac{q(t, x+\delta x) - 2q(t, x) + q(t, x-\delta x)}{\delta x^2}$$

A smoothing operation

$$q(t+\delta t, x) = q(t, x) + \frac{2k\delta t}{\delta x^2}(0.5(q(t, x+\delta x) + q(t, x-\delta x)) - q(t, x))$$

Replace q at x by the average of the values on either side